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Supercoherent State and Inhomogeneous Differential Realization of *OSP*(2, 1) Superalgebra

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The supercoherent state of *OSP*(2, 1) superalgebra is constructed and its properties are discussed in detail. The matrix elements of the *OSP*(2, 1) generators in the supercoherent state space are calculated. New inhomogeneous differential realizations of *OSP*(2, 1) superalgebra are given.

1. INTRODUCTION

Coherent states of Lie (super)algebras have played an important role in the study of quantum mechanics, quantum electrodynamics, quantum optics, and quantum field theory, and provide a natural link between classical and quantum phenomena and are related to the path integral formalism [1–7]. Recently, much attention has been paid to the coherent states of Lie (super)algebras [6–10]. Recently discovered quasi-exactly solvable problems (QESP) in quantum mechanics have become increasingly important because they have been generalized to study the conformal field theory [10]. A connection of QESP and finite-dimensional inhomogeneous differential realizations of Lie algebras (or superalgebras) has been described by Turbiner [11–14], who gave a complete classification of the one-dimensional QESP by making use of the inhomogeneous differential realization of the *SU*(2) algebra, and pointed out that the multidimensional QESP may be studied. A general procedure to construct the multidimensional QESP in terms of the inhomogeneous differential realizations of the Lie superalgebras was presented [11–15]. The key to the settlement of the QESP lies in studing finite-dimensional inhomogeneous differential realizations of Lie (super)algebras. Therefore it

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is very important to study the inhomogeneous differential realizations of Lie superalgebras. The purpose of the present paper is to derive further the new inhomogeneous differential realizations of $OSP(2, 1)$ superalgebra on the basis of studing the supercoherent state. In the present paper we shall first constuct the supercoherent state of the $OSP(2, 1)$ superalgebra and discuss its properties. Then we calculate the matrix elements of the $OSP(2, 1)$ generators in the supercoherent state representation and give new forms of the inhomogeneous differential realizations of $OSP(2, 1)$ in the supercoherent-state space

2. THE *OSP*(2, 1) SUPERCOHERENT STATE AND PROPERTIES

In accordance with the ref. 16, the generators of $OSP(2, 1)$ superalgebra are

$$\{Q_3, Q_+, Q_- \in OSP(2, 1)_{\bar{0}} | V_+, V_- \in OSP(2, 1)_{\bar{1}}\} \tag{1}$$

and satisfy the following commutation and anticommutation relations:

$$\begin{gathered} [Q_3, Q_\pm] = \pm 2Q_\pm, \qquad [Q_+, Q_-] = Q_3, \qquad [Q_3, V_\pm] = \pm V_\pm \\ [Q_\pm, V_\mp] = V_\pm, \qquad [Q_\pm, V_\pm] = 0 \\ \{V_\pm, V_\mp\} = -\tfrac{1}{4}Q_3, \qquad \{V_\pm, V_\pm\} = \pm\tfrac{1}{2}Q_\pm \end{gathered} \tag{2}$$

According to the ref. 17 and relabeling the basis vector $\phi(k, \alpha)$ of the finite-dimensional irreducible representation of $OSP(2, 1)$ superalgebra by $|N, k, \alpha\rangle$, the actions of the generators on the basis vectors are

$$\begin{aligned} Q_3|N, k, \alpha\rangle &= (-N + 2k + \alpha)|N, k, \alpha\rangle \\ Q_+|N, k, \alpha\rangle &= (N - k - \alpha)|N, k + 1, \alpha\rangle \\ Q_-|N, k, \alpha\rangle &= k|N, k - 1, \alpha\rangle \\ V_+|N, k, \alpha\rangle &= (-\tfrac{1}{2}N + \tfrac{1}{2}k)(1 - \alpha)|N, k, \alpha + 1\rangle - \tfrac{1}{2}\alpha|N, k + 1, \alpha - 1\rangle \\ V_-|N, k, \alpha\rangle &= \tfrac{1}{2}k(1 - \alpha)|N, k - 1, \alpha + 1\rangle - \tfrac{1}{2}\alpha|N, k, \alpha - 1\rangle \end{aligned} \tag{3}$$

where

$$\{|N, k, \alpha\rangle | k + \alpha \leq N, N \in Z^+, k = 0, 1, 2, \ldots, \alpha = 0, 1\}$$

and

$$k = \begin{cases} 0, 1, \ldots, N & \text{when } \alpha = 0 \\ 0, 1, \ldots, N - 1 & \text{when } \alpha = 1 \end{cases} \tag{4}$$

The space $\{|N, k, \alpha\rangle\}$ of the irrep N of $OSP(2, 1)$ superalgebra is $2N + 1$ dimensional and may be divided into two subspaces $\{|N, k, 0\rangle\}$ and $\{N, k, 1\rangle\}$ corresponding to $\alpha = 0, 1$, respectively. All the basis vectors $|N, k, \alpha\rangle$ are assumed to be normalized as

$$\binom{N}{k}\langle N, k, 0|N, k, 0\rangle = 1, \qquad \binom{N-1}{k}\langle N, k, 1|N, k, 1\rangle = 1 \tag{5}$$

The completeness condition of the vectors of the irrep may be expressed as

$$\sum_{k=0}^{N}\binom{N}{k}|N, k, 0\rangle\langle N, k, 0| + \sum_{k=0}^{N-1}\binom{N-1}{k}|N, k, 1\rangle\langle N, k, 1| = I \tag{6}$$

where I is the identity operator.

One can easily show the following formulas from (3):

$$Q_+^n|N, 0, 0, 0\rangle = \binom{N}{n}n!|N, n, 0\rangle, \qquad Q_+^n|N, 0, 1\rangle = \binom{N-1}{n}n!|N, n, 1\rangle \tag{7}$$

where

$$\binom{N}{n} = \frac{N!}{(N-n)!n!}$$

In terms of Bloch's method we now define the supercoherent state $|Z, \xi\rangle$ by applying the exponential operator $\exp(ZQ_+ + \xi V_+)$ on the lowest weight state $|N, 0, 0\rangle$ of the $OSP(2, 1)$ irrep,

$$|Z, \xi\rangle = \exp(ZQ_+ + \xi V_+)|N, 0, 0\rangle \tag{8}$$

where Z and ξ are one complex variable and one Grassmann variable, respectively. Considering the generator Q_+ as commutable with V_+,

$$[Q_+, V_+] = 0 \tag{9}$$

we can easily show the following formula:

$$\exp(ZQ_+ + \xi V_+) = \exp(ZQ_+)\exp(\xi V_+) \tag{10}$$

Using the formula Eqs. (7) and (10), we can rewrite the supercoherent state equation (8) as follows:

$$|Z, \xi\rangle = \sum_{n=0}^{N} \binom{N}{n} Z^n |N, n, 0\rangle - \frac{1}{2} N\xi \sum_{n=0}^{N-1} \binom{N-1}{n} Z^n |N, n, 1\rangle$$
$$= |Z\rangle_1 - \frac{1}{2} N\xi |Z\rangle_2 \tag{11}$$

where $|Z\rangle_1$and $|Z\rangle_2$ are two simple coherent states associated with two subspaces $\{|N, k, 0\rangle\}$, $\{|N, k, 1\rangle\}$ of the $OSP(2, 1)$ irrep,

$$|Z\rangle_1 = \sum_{n=0}^{N} \binom{N}{n} Z^n |N, n, 0\rangle, \qquad |Z\rangle_2 = \sum_{n=0}^{N-1} \binom{N-1}{n} Z^n |N, n, 1\rangle \tag{12}$$

According to Eq. (11), we have

$$\langle Z, \xi| = {}_1\langle Z| - \tfrac{1}{2} N {}_2\langle Z| \bar{\xi} \tag{13}$$

where $\bar{\xi}$ is the complex conjugation of ξ.

We may write the scalar product of two such states as ${}_i\langle Z'|Z\rangle_i$. We see from Eq. (12) that these scalar products are

$$\begin{aligned} &{}_1\langle Z'|Z\rangle_1 = (1 + \bar{Z}'Z)^N, \qquad {}_2\langle Z'|Z\rangle_2 = (1 + \bar{Z}'Z)^{N-1} \\ &{}_i\langle Z'|Z\rangle_j = 0, \qquad\qquad i \neq j, \quad i, j = 1, 2 \end{aligned} \tag{14}$$

which means that the two simple coherent states with different Z in the same subspace are not orthogonal to each other. Nevertheless, two coherent states in different subspaces are orthogonal to each other.

Similarly, the scalar product of the supercoherent state is written as follows:

$$\langle Z', \xi|Z, \xi\rangle = (1 + \bar{Z}'Z + \tfrac{1}{4} N^2 \bar{\xi}'\xi)(1 + \bar{Z}'Z)^{N-1} \tag{15}$$

Putting $Z' = Z$, $\xi' = \xi$ in Eq. (15), we can write the orthogonality relation of the supercoherent state $|Z, \xi\rangle$,

$$\langle Z, \xi|Z, \xi\rangle = (1 + \bar{Z}Z + \tfrac{1}{4} N^2 \bar{\xi}\xi)(1 + \bar{Z}Z)^{N-1} \tag{16}$$

The expansion coefficients of the supercoherent state $|Z, \xi\rangle$ can be found in terms of the complete orthonormal set $\{|N, k, \alpha\rangle\}$. Thus, we have

$$\langle Z, \xi|N, k, 0\rangle = \bar{Z}^k, \qquad \langle Z, \xi|N, k, 1\rangle = \tfrac{1}{2} N \bar{\xi} \bar{Z}^k \tag{17}$$

While orthogonality is a convenient property for a set of basis vectors, it is not a necessary one. The essential property of such a set is that it be complete. Since the $2N + 1$ state vectors $\{|N, k, \alpha\rangle\}$ of an irrep of $OSP(2, 1)$ superalgebra are known to form a completeness orthogonal set, the supercoherent state $|Z, \xi\rangle$ for $OSP(2, 1)$ superalgebra can be shown without difficulty to form a complete set. To give a proof we need only demonstrate that the unit operator may be expressed as a suitable sum or an integral, over the superplane, of projection operators of the form $|Z, \xi\rangle\langle Z, \xi|$. In order to describe such an integral we introduce generally the differential element of weight area in the superplane,

$$d^2Z\, d^2\xi\, \sigma(Z, \xi) = |Z|\, d|Z|\, d\theta\, d\bar{\xi}\, d\xi\, \sigma(Z, \xi) \tag{18}$$

where $\sigma(Z, \xi)$ is a weight superfield function and $Z = |Z|e^{i\theta}$.

The problem here may by changed to find the weight superfield function $\sigma(Z, \xi)$ such that

$$\int d^2Z\, d^2\xi\, \sigma(Z, \xi)|Z, \xi\rangle\langle Z, \xi|$$
$$= \sum_{k=0}^{N} \binom{N}{k} |N, k, 0\rangle\langle N, k, 0| + \sum_{k=0}^{N-1} \binom{N-1}{k} |N, k, 1\rangle\langle N, k, 1| = 1 \tag{19}$$

where $d^2Z = |Z|\, d|Z\, d\theta$, $d^2\xi = d\bar{\xi}\, d\xi$.

To determine $\sigma(Z, \xi)$, we expand $\sigma(Z, \xi)$ in ξ and save two effective items for the integral Eq. (19), i.e.,

$$\sigma(Z, \xi) = A(Z) + B(Z)\bar{\xi}\xi \tag{20}$$

where $A(Z)$ and $B(Z)$ are two expansion coefficients. Substituting the definition of simple coherent state (12) into Eq. (19) and integrating over the entire area of the superplane, we have

$$\int d^2Z\, d^2\xi\, \sigma(Z, \xi)|Z, \xi\rangle\langle Z, \xi|$$
$$= \int d^2Z\, B(Z)|Z\rangle_{11}\langle Z| + \frac{1}{4} N^2 \int d^2Z\, A(Z)|Z\rangle_{22}\langle Z|$$
$$= 2\pi \sum_{n=0}^{N} \binom{N}{n}\binom{N}{n} \int_0^{\infty} B(Z)|Z|^{2n+1}\, d|Z|\, |N, n, 0\rangle\langle N, n, 0|$$
$$+ 2\pi \sum_{n=0}^{N-1} \frac{1}{4} N^2 \binom{N-1}{n}\binom{N-1}{n}$$
$$\times \int_0^{\infty} A(Z)|Z|^{2n+1}\, d|Z|\, |N, n, 1\rangle\langle N, n, 1| = 1 \tag{21}$$

In calculating the integral (21) we used the Grassmann integral

$$\int d\xi = \int d\bar{\xi} = 0, \qquad \int \xi\, d\xi = \int \bar{\xi}\, d\, \bar{\xi} = 1 \tag{22}$$

Comparing Eq. (21) with Eq. (6), we must have

$$2\pi \binom{N}{n} \int_0^\infty B(Z)|Z|^{2n+1}\, d|Z| = 1,$$
$$2\pi \frac{1}{4} N^2 \binom{N-1}{n} \int_0^\infty A(Z)|Z|^{2n+1}\, d|Z| = 1 \tag{23}$$

With the aid of the integral identity

$$\int_0^\infty \frac{x^{2n+1}}{(1+x^2)^m}\, dx = \frac{n!\,(m-n-2)!}{2(m-1)!} \tag{24}$$

and by comparing Eqs. (23) with Eq. (24), we obtain the following expansion coefficients:

$$B(Z) = \frac{N+1}{\pi(1+\bar{Z}Z)^{N+2}}, \qquad A(Z) = \frac{4}{\pi N(1+\bar{Z}Z)^{N+1}} \tag{25}$$

Substituting the above expansion coefficients into Eq. (20), we finally obtain the weight superfield function:

$$o(Z, \xi) = \frac{1}{\pi}\left[(N+1)\bar{\xi}\xi + \frac{4}{N}(1+\bar{Z}Z)\right](1+\bar{Z}Z)^{-N-2}$$

We have thus shown

$$\frac{1}{\pi}\int d^2Z\, d^2\xi\Big[(N+1)\bar{\xi}\xi + \frac{4}{N}(1+\bar{Z}Z)\Big](1+\bar{Z}Z)^{-N-2}|Z, \xi\rangle\langle Z, \xi| = 1 \tag{26}$$

which is a completeness relation for the supercoherent state of *OSP*(2, 1) superalgebra of precisely the type desired. As a result of the above completeness relation, an arbitrary vector $|\Psi\rangle$ can be expanded in terms of the supercoherent state for *OSP*(2, 1) superalgebra. To secure the expansion of $|\Psi\rangle$ in terms of the supercoherent state $|Z, \xi\rangle$, we multiply $|\Psi\rangle$ by the representation (26) of the unit operator. We then find

$$|\psi\rangle = \frac{1}{\pi}\int d^2Z\, d^2\xi\, \Big[(N+1)\bar{\xi}\xi + \frac{4}{N}(1+\bar{Z}Z)\Big](1+\bar{Z}Z)^{-N-2}|Z, \xi\rangle\langle Z, \xi|\psi\rangle \tag{27}$$

3. MATRIX ELEMENTS OF THE *OSP*(2, 1) GENERATORS

The present section will be devoted to calculating the matrix elements of the *SPL*(2, 1) generators in the supercoherent-state representation. The results are as follows:

$$\begin{aligned}
\langle Z', \xi' | Q_3 | Z, \xi\rangle &= -N[(1+\bar{Z}'Z) + \tfrac{1}{4}N(N-1)\bar{\xi}'\xi](1-\bar{Z}'Z)(1+\bar{Z}'Z)^{N-2} \\
\langle Z', \xi' | Q_+ | Z, \xi\rangle &= N[1+\bar{Z}'Z + \tfrac{1}{4}N(N-1)\bar{\xi}'\xi\bar{Z}'(1+\bar{Z}'Z)^{N-2} \\
\langle Z', \xi' | Q_- | Z, \xi\rangle &= N[1+\bar{Z}'Z + \tfrac{1}{4}N(N-1)\bar{\xi}'\xi\bar{Z}(1+\bar{Z}'Z)^{N-2} \\
\langle Z', \xi' | V_+ | Z, \xi\rangle &= \tfrac{1}{4}N(\bar{Z}'\xi + \bar{\xi}')(1+\bar{Z}'Z)^{N-1} \\
\langle Z', \xi' | V_- | Z, \xi\rangle &= \tfrac{1}{4}N[\xi - (N-1)Z\bar{\xi}'](1+\bar{Z}'Z)^{N-1}
\end{aligned} \tag{28}$$

In evaluating the matrix elements, one needs use Eqs. (3), (5), and (12), for example,

$$\begin{aligned}
&\langle Z', \xi' | Q_3 | Z, \xi\rangle \\
&\quad = \langle Z', \xi' | Q_3 \sum_{n=0}^{N} \binom{N}{n} |N, n, 0\rangle\langle N, n, 0| + \sum_{n=0}^{N-1} \binom{N-1}{n} |N, n, 1\rangle\langle N, n, 1 | Z, \xi\rangle \\
&\quad = \sum_{n=0}^{N} (-N+2n)\binom{N}{n}\langle Z', \xi' | N, n, 0\rangle\langle N, n, 0 | Z, \xi\rangle \\
&\qquad + \sum_{n=0}^{N-1} (-N+2n+1)\binom{N-1}{n}\langle Z', \xi' | N, n, 1\rangle\langle N, n, 1 | Z, \xi\rangle \\
&\quad = \sum_{n=0}^{N} (-N+2n)\binom{N}{n}(\bar{Z}'Z)^n + \sum_{n=0}^{N-1} (-N+2n+1)\binom{N-1}{n}\frac{N^2}{4}\bar{\xi}'\xi(\bar{Z}'Z)^n \\
&\quad = -N[(1+\bar{Z}'Z) + \frac{1}{4}N(N-1)\bar{\xi}'\xi](1-\bar{Z}'Z)(1+\bar{Z}'Z)^{N-2}
\end{aligned} \tag{29}$$

4. NEW INHOMOGENEOUS DIFFERENTIAL REALIZATION OF *OSP*(2, 1)

We now consider the actions of the *OSP*(2,1) generators on the supercoherent state $|Z, \xi\rangle$, i.e.,

$$G|Z, \xi\rangle = D(G)|Z, \xi\rangle \tag{30}$$

where G stands for the *OSP*(2, 1) generators and $D(G)$ the realizations of the generators.

To find explicit forms for the generator realizations, we begin by considering the relations between the simple coherent states and the supercoherent state. Using the definition of the supercoherent state (11), we find

$$|Z\rangle_1 = \left(1 - \xi \frac{\partial}{\partial \xi}\right)|Z, \xi\rangle$$
$$|Z\rangle_2 = -\frac{2}{N}\frac{\partial}{\partial \xi}\,|Z, \xi\rangle \tag{31}$$

By making use of Eqs. (11) and (31), we can construct explicitly the differential realizations $D(G)$ of the $OSP(2, 1)$ generators as follows:

$$D(Q_3) = -N + \xi \frac{\partial}{\partial \xi} + 2Z \frac{\partial}{\partial Z}$$
$$D(Q_+) = \frac{\partial}{\partial Z}, \qquad D(Q_-) = NZ - Z\xi \frac{\partial}{\partial \xi} - Z^2 \frac{\partial}{\partial Z} \tag{32}$$
$$D(V_+) = \frac{\partial}{\partial \xi} + \frac{1}{4}\xi \frac{\partial}{\partial Z}, \qquad D(V_-) = \frac{1}{4}N\xi - Z\frac{\partial}{\partial \xi} - \frac{1}{4}Z\xi \frac{\partial}{\partial Z}$$

For example, for $D(V_-)$, we have

$$\begin{aligned} V_-|Z, \xi\rangle &= V_-\left[|Z_1 - \frac{1}{2}N\xi|Z\rangle_2\right] \\ &= \frac{1}{2}NZ|Z\rangle_2 + \frac{1}{2}N\xi\frac{1}{2}\frac{1}{N}\left(N - Z\frac{\partial}{\partial Z}\right)|Z\rangle_1 \\ &= \frac{1}{2}NZ\left(-\frac{2}{N}\frac{\partial}{\partial \xi}\right)|Z, \xi\rangle + \frac{1}{4}\xi\left(N - Z\frac{\partial}{\partial Z}\right)\left(1 - \xi\frac{\partial}{\partial \xi}\right)|Z, \xi\rangle \\ &= \left(\frac{1}{4}N\xi - Z\frac{\partial}{\partial \xi} - \frac{1}{4}Z\xi\frac{\partial}{\partial Z}\right)|Z, \xi\rangle \end{aligned} \tag{33}$$

It is clear that the aforesaid realizations are inhomogeneous. Therefore, they may be of use for quasi-exactly solvable problems in quantum mechanics.

We have constructed the supercoherent state of $OSP(2, 1)$ superalgebra. We also have calculated the matrix elements of the $OSP(2, 1)$ generators. The new inhomogeneous differential realizations of the $OSP(2, 1)$ generators have been obtained in the supercoherent-state space.

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